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SELF-SIMILAR SOLUTIONS OF UNSTEADY BOUNDARY LAYERS

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Self-similar solutions are considered for the unsteady dynamic-diffusion boundary layer that forms near a vertical wall at high Schmidt numbers and for the dynamic boundary layer adjacent to the dynamic-diffusion layer at the inner edge. It is shown that a countercurrent flow zone forms in the flow region of the dynamic boundary layer.

Introduction. Free convection in a viscous fluid near a vertical wall and transfer of an impurity are described employing the classical Oberbeck–Boussinesq model and the model of microconvection. It is known that using the Oberbeck–Boussinesq model at high Reynolds numbers, one can distinguish a boundary layer and obtain integral flow characteristics (Nusselt numbers) from the solutions of the problem. In the case of microconvection, the Reynolds numbers are usually low. Kuznetsov and Frolovskaya [1] proposed a method for distinguishing a dynamic-diffusion boundary layer in the case of microconvection, where the Oberbeck–Boussinesq model is inapplicable. In both models, a special dynamic-diffusion boundary layer is distinguished at high Schmidt (Prandtl) numbers with no restrictions imposed on the Reynolds number. In these boundary layers, the viscous and buoyancy forces were substantial and the inertial forces were negligible. Outside the dynamic-diffusion boundary layer, the structure of the velocity field depends on Reynolds number. If the Reynolds number is high, in the flow region there is one more purely dynamic layer of greater asymptotic thickness, whose inner edge is adjacent to the dynamic-diffusion layer and whose outer edge neighbors the state of rest.

Kuznetsov and Frolovskaya [1] formulated the equations of steady-state dynamic-diffusion boundary layer, found self-similar solutions of these equations, and considered their initial asymptotic forms. Results of studies of free convective flows are given in greater detail in [2, 3].

Unsteady Boundary Layers. We consider the problem of determining the u and v components of the velocity $\boldsymbol{v}$, the concentration c, and the deviation from the hydrostatic pressure p in a region $y > 0$ bounded by an infinite vertical wall $\{y = 0\}$. The gravity force is directed along the Ox axis. In the (x, y) coordinates, the acceleration of gravity is written as $\boldsymbol{g} = (-g, 0)$. We assume that the density of the melt ρ depends linearly on the concentration: $\rho = \rho_0[1 + \beta(c - c_\infty)]$, where ρ_0 and c_∞ are the average density and concentration of the solution, respectively, and $\beta = (1/\rho_0)(d\rho/dc) = \text{const}$ [for definiteness, we set $\beta > 0$)].

At high Schmidt numbers $\text{Sc} = \nu/D$ near the vertical wall, using the same assumptions as in [1], we can distinguish an unsteady dynamic-diffusion layer. For the Boussinesq model, the boundary-layer equations are

$$\nu \frac{\partial^2 u}{\partial y^2} = g\beta(c - c_\infty), \qquad \frac{1}{\rho_0}\frac{\partial p}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2}. \tag{1}$$

The boundary conditions for the velocity are given by

$$u|_{y=0} = v|_{y=0} = 0, \qquad u \underset{y\to\infty}{\longrightarrow} u_\infty(t, x). \tag{2}$$

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For the concentration, we specify the first-order boundary conditions

$$c|_{t=0} = c_\infty, \qquad c|_{y=0} = f(t,x), \qquad c \underset{y\to\infty}{\longrightarrow} c_\infty \tag{3}$$

or the second-order boundary conditions

$$c|_{t=0} = c_\infty, \qquad \frac{\partial c}{\partial y}\Big|_{y=0} = h(t,x), \qquad c \underset{y\to\infty}{\longrightarrow} c_\infty, \tag{4}$$

where $c_\infty = \mathrm{const}$, $f(t,x)$ and $h(t,x)$ are specified functions, and $u_\infty(t,x)$ is determined in the course of solution of the problem.

The problem (1)–(3) [or (1), (2), and (4)] describes the motion in a thin dynamic-diffusion layer with thickness of order $(\mathrm{ScRe}^2)^{-1/4}$; outside the layer $c \approx c_\infty$. In this layer, the buoyancy and viscous forces are of the same order of magnitude, and, the inertial forces and the longitudinal pressure gradient are negligible in comparison with them. Unlike in the case of a classical boundary layer [2], the external representation for the velocity is determined during solution and not from the condition of joining with the external solution:

$$u = v = 0, \qquad p = 0, \qquad c = c_\infty. \tag{5}$$

The velocity components are determined independently of the pressure, which is obtained by integration of the second of Eqs. (1) over y from y to ∞ subject to the continuity equation

$$p(t,x,y) = p_\infty(t,x) + \rho_0\nu\Big(\frac{\partial u_\infty}{\partial x} - \frac{\partial u}{\partial x}\Big), \tag{6}$$

where $p_\infty(t,x)$ is the pressure at the outer edge of the boundary layer.

For the dynamic-diffusion boundary layer in the case of microconvection, it is required to determine the concentration c, the modified velocity $\boldsymbol{w} = \boldsymbol{v} + \beta D\nabla c$, and the modified pressure $q = p/\rho_* - gx + \beta(\nu - D)D\Delta c$ with $\rho = \rho_*(1 - \beta(c - c_\infty))^{-1}$ that satisfy the initial boundary-value problem [1]

$$\begin{gathered}
\nu(1-\beta(c-c_\infty))\frac{\partial^2 w_1}{\partial y^2} = g\beta(c-c_\infty), \qquad \frac{\partial q}{\partial y} = \nu\frac{\partial^2 w_2}{\partial y^2}, \qquad \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0,\\
\frac{\partial c}{\partial t} + w_1\frac{\partial c}{\partial x} + w_2\frac{\partial c}{\partial y} - \beta D\Big(\frac{\partial c}{\partial y}\Big)^2 = D(1-\beta(c-c_\infty))\frac{\partial^2 c}{\partial y^2},\\
w_1|_{y=0} = \beta D\frac{\partial c}{\partial x}\Big|_{y=0}, \qquad w_2|_{y=0} = \beta D\frac{\partial c}{\partial y}\Big|_{y=0}, \qquad w_1 \underset{y\to\infty}{\longrightarrow} w_\infty(t,x) < \infty,\\
c|_{t=0} = c_\infty, \qquad c|_{y=0} = r(t,x), \qquad c \underset{y\to\infty}{\longrightarrow} c_\infty.
\end{gathered} \tag{7}$$

Here w_1 and w_2 are the velocity components $\boldsymbol{w}$, $r(t,x)$ is a specified function, and $w_\infty(t,x)$ is determined during the solution of the problem. Here for the concentration, we can also specify conditions of the first kind.

Since generally, $u_\infty \neq 0$ ($w_\infty \neq 0$), the solution of the problem (1)–(3) [or (7)] cannot be joined with the external solution (5). To eliminate this discrepancy for $\mathrm{Sc}/\mathrm{Re}^2 \to 0$, as in [1], we can distinguish one more asymptotic form of the problem that describes motion in a region with asymptotic thickness greater than the boundary-layer thickness considered above. In this case, Prandtl's hypothesis on the equality of the orders of magnitude of viscous and inertial forces is valid. The motion in this layer of thickness $(\mathrm{Sc}/\mathrm{Re}^2)^{1/4}$ is described by the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8}$$

At the initial time,

$$u|_{t=0} = 0. \tag{9}$$

From the joining conditions, we obtain the following boundary conditions for the longitudinal velocity component u:

$$u|_{y=0} = u_\infty(t,x), \qquad u \underset{y\to\infty}{\longrightarrow} 0. \tag{10}$$

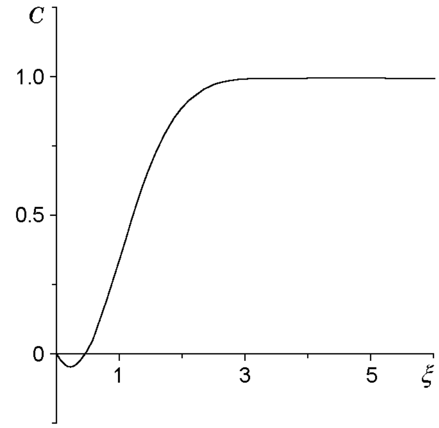

Fig. 1

The boundary condition for the transverse velocity component v is given by

$$v|_{y=0} = 0. \tag{11}$$

The dynamic-layer problem differs from the classical problem in that the longitudinal velocity is specified at the inner rather than at the outer edge. In this problem, the pressure can be considered vanishing because, as follows from the second of Eqs. (8), the pressure p is the same as the pressure at $y \to \infty$, where $p \equiv 0$ (state of rest; the pressure is equal to hydrostatic pressure). Therefore, in this case, $p_\infty \equiv 0$ in formula (6).

Self-Similar Solutions. If conditions of the first kind are specified, we can seek self-similar solutions of the problem (1)–(3) for $f(t,x) = c_\infty - \nu x t^{-2}/(g\beta D)$. We will seek a solution of the problem considered in the form $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, and $c - c_\infty = (c_\infty - f(t,x))(C(\xi) - 1)$, where the stream function is written as

$$\psi = \sqrt{D} x t^{-1/2} \Psi(\xi), \qquad \xi = y t^{-1/2}/\sqrt{D}.$$

Then, Eqs. (1) become

$$\Psi''' = C - 1, \qquad C'' = (\Psi' - 2)(C - 1) - (\Psi + \xi/2)C'. \tag{12}$$

From conditions (2) and (3), it follows that

$$\Psi(0) = \Psi'(0) = 0, \quad C(0) = 0, \quad \lim_{\xi\to\infty} \Psi'(\xi) = U_\infty = \text{const} < \infty, \quad \lim_{\xi\to\infty} C(\xi) = 1. \tag{13}$$

Here the initial conditions are not specified. A typical concentration profile is shown in Fig. 1.

For external representation of the velocity, we have

$$u_\infty(x) = U_\infty x t^{-1} \approx 0.975 x t^{-1}.$$

To characterize the mass transfer between the growing film and the solution, we introduce the overall and local Nusselt numbers:

$$\mathrm{Nu} = \int_0^l \frac{1}{c_\infty - c_\omega} \frac{\partial c}{\partial y}\Big|_{y=0} dx, \qquad \mathrm{Nu}_x = \frac{x}{c_\infty - c_\omega} \frac{\partial c}{\partial y}\Big|_{y=0},$$

where c_ω is the concentration on the wall $\{y = 0\}$. For the solutions considered, the formulas for the Nusselt numbers become

$$\mathrm{Nu} = \frac{l t^{-1/2}}{\sqrt{D}} |C'(0)| \approx 0.434 \frac{l t^{-1/2}}{\sqrt{D}}, \qquad \mathrm{Nu}_x = \frac{x t^{-1/2}}{\sqrt{D}} |C'(0)| \approx 0.434 \frac{x t^{-1/2}}{\sqrt{D}}.$$

We determine the thickness of the dynamic-diffusion layer. In classical theory, the boundary-layer thickness is evaluated using the so-called displacement thickness [4]. In our case, a characteristic feature of the boundary

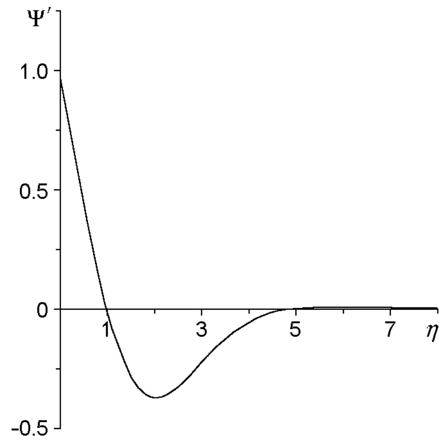

Fig. 2

layer is that the concentration c inside the layer differs from the average value, and outside the layer, $c \approx c_\infty$. An analog of the displacement thickness δ_c^* is defined by the equality

$$\delta_c^*(c_\infty - c_\omega) = \int_0^\infty [c_\infty - c(t,x,y)]\, dy.$$

Calculations for self-similar solutions yield

$$\delta_c^* = \sqrt{Dt} \int_0^\infty [1 - C(\xi)]\, d\xi \approx 1.276\sqrt{Dt}.$$

In the case of conditions of the second kind specified for the concentration on the wall, a self-similar solution of the problem (1), (2), (4) can be derived if

$$h(t,x) = q\nu x t^{-5/2}/(g\beta D\sqrt{D}) \qquad (q = \text{const} \geqslant 0).$$

In this case, we seek a solution of the same form as that for the problem of the first kind. Equations (12) and boundary conditions (13) remain unchanged except for the wall concentration condition: in (13), the condition $C(0) = 0$ is replaced by $C'(0) = q$. Depending on the heat flux (for $0 < q < 1$), the wall concentration varies as

$$c_\infty - c|_{y=0} = \nu x t^{-2}(1 - C(0))/(g\beta D),$$

where $1 - C(0) = 0.139q^3 - 0.485q^2 + 1.896q + 2.132$.

The problem (7) does not admit a self-similar solution.

We seek a solution of the problem (8)–(11) in the form $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, where the stream function ψ is written as

$$\psi = \sqrt{\nu} x t^{-1/2}\Psi(\eta), \qquad \eta = y t^{-1/2}/\sqrt{\nu}.$$

Then, to determine Ψ, we have the problem

$$\Psi''' = (\Psi' - 1)\Psi' - (\Psi + \eta/2)\Psi'', \qquad \Psi(0) = 0, \quad \Psi'(0) = U_\infty, \quad \lim_{\eta\to\infty} \Psi'(\eta) = 0. \tag{14}$$

Numerical solution of the problem (14) shows that a countercurrent flow zone forms in the flow region. A curve of $\Psi'(\eta)$ is presented in Fig. 2. In this case (unlike in the classical case), we can calculate the volume flow Q in the dynamic boundary layer:

$$Q = \int_0^\infty u(t,x,y)\, dy.$$

The displacement thickness of the dynamic layer is obtained from the formula

$$\delta_v^* u_\infty(t,x) = \int_0^\infty u(t,x,y)\,dy.$$

Calculations yield $Q = \sqrt{\nu} x t^{-1/2} \Psi_\infty \approx -0.256 \sqrt{\nu} x t^{-1/2}$. The thickness of the countercurrent flow region

$$\delta = ((\Psi(\eta_*) - |\Psi_\infty|)/U_\infty)\sqrt{\nu t} \approx 0.201\sqrt{\nu t}$$

is 76.66% of the dynamic-layer thickness δ_v^*. Here Ψ_∞ is the value of $\Psi(\eta)$ for $\eta \to \infty$, and η_* is a point at which $\Psi' = 0$.

Conclusions. The problem of mass transfer and free convection near a vertical wall at high Schmidt numbers was considered. For the unsteady flow regime, self-similar solutions are derived. Formulas for mass transfer are obtained.

If the Reynolds number is high, in the flow region there is also a purely dynamic boundary layer of greater asymptotic thickness, whose inner edge is adjacent to the dynamic-diffusion layer. A countercurrent flow zone occurs in the flow region.

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REFERENCES

1. V. V. Kuznetsov and O. A. Frolovskaya, “Boundary layers in free convection,” *Prikl. Mekh. Tekh. Fiz.*, **41**, No. 3, 92–100 (2000).
2. O. G. Martynenko and Yu. A. Sokovishin, *Free-Convective Heat Transfer on a Vertical Surface* [in Russian], Nauka Tekhnika, Minsk (1977).
3. O. G. Martynenko, A. A. Berezovskii, and Yu. A. Sokovishin, *Asymptotic Methods in the Theory of Free-Convective Heat Transfer* [in Russian], Nauka Tekhnika, Minsk (1979).
4. G. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York (1968).